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Edge states for quantum Hall droplets in higher dimensions and a generalized WZW model

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Abstract

We consider quantum Hall droplets on complex projective spaces with a combination of abelian and nonabelian background magnetic fields. Carrying out an analysis similar to what was done for abelian backgrounds, we show that the effective action for the edge excitations is given by a chiral, gauged Wess-Zumino-Witten (WZW) theory generalized to higher dimensions.

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1 Introduction

The study of quantum Hall effect in dimensions higher than two and especially the edge excitations of quantum Hall droplets have been of some research interest following the original analysis by Zhang and Hu [1]. They considered the Landau problem for charged fermions on S^4 with a background magnetic field corresponding to the standard $SU(2)$ instanton. In this case, the edge excitations, which correspond to the deformations of the edge of a quantum Hall droplet, are gapless, as in two dimensions, but now have spin degrees of freedom as well. In particular, one could have spin-two gapless excitations suggesting an alternate description of the graviton. Although this has not been born out in the expected fashion, edge excitations for a quantum Hall droplet lead to a very interesting class of field theories. The original model has by now been extended to many new geometries and contexts [2]- [4]. In a recent paper, we gave a general method for obtaining the effective action for the edge excitations of a quantum Hall droplet [5]. This was applied to higher dimensional quantum Hall systems, specifically for droplets in the complex projective space $\mathbf{CP}^k$. On $\mathbf{CP}^k$ one can have abelian or nonabelian background magnetic fields. In the case of an abelian background, which was analyzed in [5], the edge excitations can be described in terms of a chiral action for an abelian bosonic field. For $\mathbf{CP}^1 = S^2$, this is the well known one-dimensional chiral bosonic action, defined on the boundary S^1, which describes the edge excitations of the $\nu = 1$ two-dimensional QHE [6]. For general even dimensional spaces $\mathbf{CP}^k$, the notion of chirality is expressed by the fact that there is a preferred direction along the S^{2k-1} droplet boundary which is relevant for the dynamics of the edge excitations. This is determined by the Kähler form of $\mathbf{CP}^k$. The effective action is eventually integrated over all tangential directions. The bosonic field itself can be expanded in terms of the boundary surface oscillation modes of the incompressible droplet. This analysis also leads to the edge action for QHE droplets on S^4, using the fact that $\mathbf{CP}^3$ is an S^2-bundle over S^4.

In this paper we extend our analysis of the edge effective action in the case of nonabelian background magnetic fields, thus completing the full derivation of the edge action for all backgrounds. [2] In this case the edge excitations are described in terms of a matrix valued bosonic field and the corresponding effective action is given in terms of a higher dimensional generalization of a nonabelian, chiral, gauged Wess-Zumino-Witten action. The form of the action suggests some kind of scale invariance and the possibility of an elegant current algebra structure. Therefore we believe these actions define an interesting new class of theories in higher dimensions which are worth studying in their own right.

This paper is organized as follows. In section 2, we review briefly the results for the edge effective action for quantum Hall droplets in $\mathbf{CP}^k$ with a $U(1)$ background field and state the main result that we derive for the nonabelian case. In section 3 we present a detailed analysis when the background field is $U(k)$. The effective action for the edge excitations is derived in the limit where the number of states of the lowest Landau level, N, becomes very large. We start with a discussion of the lowest Landau level (LLL) wavefunctions. We then discuss star products and evaluate the large N limit of operator commutators in terms of symbols, which are classical functions corresponding to operators. Using this we derive the large N limit of the kinetic energy

[2] It may be worth noting that a particular case of the action for edge states with a nonabelian background was already considered in the context of S^3 in [4].

term of the action. We separately evaluate the symbol corresponding to the density operator and using this we simplify the potential energy term. We finally combine everything to arrive at the effective action which is given in equation (9). This action is a higher dimensional generalization of a chiral, gauged Wess-Zumino-Witten (WZW) theory. In section 4 we discuss in more detail properties of this effective action and we point out similarities to other higher dimensional WZW models, such as the Kähler WZW models [7]. The paper is concluded with a short discussion.

2 The effective action for edge states

The effective action for edge states was obtained in [5] by considering fluctuations of the density matrix following the strategy used by Sakita [8] in the case of the two-dimensional droplet. We shall begin by briefly recalling the essential points of this method. Let $\hat{\rho}_0$ be the density matrix corresponding to the droplet. All elements of $\hat{\rho}_0$ are zero except for the diagonal ones corresponding to the filled states, for which it is equal to one. Time evolution of $\hat{\rho}_0$ is given by a unitary matrix $\hat{U}$. The action which determines $\hat{U}$ is given by

$$S = \int dt \ \Big[i\mathrm{Tr}(\hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U}) \ - \ \mathrm{Tr}(\hat{\rho}_0 \hat{U}^\dagger \hat{\mathcal{H}} \hat{U}) \Big] \tag{1}$$

where $\hat{\mathcal{H}}$ is the Hamiltonian. We used the following strategy in simplifying this action. First of all, we restrict to the single particle Hilbert space, since it is a good approximation to neglect any dynamically correlated states of two or more fermions. This is essentially a Hartree-Fock approximation. For a droplet in the lowest Landau level (LLL) or any fixed Landau level, we can then take the Hamiltonian, up to an additive constant, as the potential $\hat{V}$ which confines the particles to the droplet. If N is the dimension of the one-particle Hilbert space and K states are occupied, we can simplify (1) by taking $N \to \infty$ and K very large but finite.

We will first recapitulate the results for an abelian background. As we have shown in [2], [5] the LLL states are characterized by an integer $n \sim BR^2$ where B is the magnetic field and R is the radius of the $\mathbf{CP}^k$; the dimension N of the Hilbert space corresponding to the LLL sector depends on n, in particular $N \sim n^k$ and scaling is done with $R \sim \sqrt{n}$. In the large N limit two main results were used to identify the edge effective action.

1. The operator commutators tend to the Poisson brackets of appropriate classical functions, namely

$$[\hat{A}, \hat{B}] \to \frac{i}{n}\{A, B\} \tag{2}$$

where $A,\ B$ are the classical functions corresponding to $\hat{A},\ \hat{B}$ and the Poisson bracket is defined by

$$\begin{aligned} \{A, B\} &= (\Omega^{-1})^{\bar{i}j}\left(\frac{\partial A}{\partial \bar{\xi}^i}\frac{\partial B}{\partial \xi^j} - \frac{\partial A}{\partial \xi^j}\frac{\partial B}{\partial \bar{\xi}^i}\right) \\ &= i(1 + \bar{\xi}\cdot\xi)\left(\frac{\partial A}{\partial \xi^i}\frac{\partial B}{\partial \bar{\xi}^i} - \frac{\partial A}{\partial \bar{\xi}^i}\frac{\partial B}{\partial \xi^i}\right. \end{aligned}$$

$$+\xi\cdot\frac{\partial A}{\partial\xi}\bar{\xi}\cdot\frac{\partial B}{\partial\bar{\xi}}-\bar{\xi}\cdot\frac{\partial A}{\partial\bar{\xi}}\xi\cdot\frac{\partial B}{\partial\xi}\Big) \tag{3}$$

Here $\xi^i=(x_i+iy_i)/R,\ \bar{\xi}^i=(x_i-iy_i)/R$ are the local complex coordinates and Ω is the Kähler two-form on $\mathbf{CP}^k$

$$\begin{aligned}\Omega &= \Omega_{i\bar{j}}d\xi^i\wedge d\bar{\xi}^j\\ &= i\Big[\frac{d\xi\wedge d\bar{\xi}}{1+\bar{\xi}\cdot\xi}-\frac{\bar{\xi}\cdot d\xi\wedge\xi\cdot d\bar{\xi}}{(1+\bar{\xi}\cdot\xi)^2}\Big]\end{aligned} \tag{4}$$

We can write $\hat{U}=\exp(i\hat{\Phi})$ for some hermitian operator $\hat{\Phi}$, expand various terms in (1) in terms of commutators and extract their large N-limit. If the background magnetic field is abelian, the classical field Φ corresponding to $\hat{\Phi}$ is a scalar which parametrizes the surface deformations of the droplet.

2. The classical function ρ_0, the symbol corresponding to the density operator $\hat{\rho}_0=\sum_{i=1}^K|i\rangle\langle i|$ is essentially a step function identifying the droplet. The term $\frac{\partial\rho_0}{\partial r^2}$, where $r^2=\bar{\xi}\cdot\xi$, is a δ-function with support only at the edge of the droplet.

Using these two results and taking into account the scaling properties of various quantities as $n\to\infty$, we found that the final effective action, up to an additive constant, is of the form

$$S=-\frac{M^{k-1}}{4\pi^k}\int_{\partial\mathcal{D}}dt\left[\frac{\partial\Phi}{\partial t}+\omega(\mathcal{L}\Phi)\right]\mathcal{L}\Phi \tag{5}$$

where M is related to the radius of the droplet, $R_D\sim\sqrt{M}$, and

$$\mathcal{L}\Phi=i\Big(\xi\cdot\frac{\partial}{\partial\xi}-\bar{\xi}\cdot\frac{\partial}{\partial\bar{\xi}}\Big)\Phi \tag{6}$$

Notice that the full effective action is an edge action defined on the boundary $\partial\mathcal{D}$ of the droplet D. $\mathcal{L}\Phi$ involves only derivatives of Φ with respect to a particular tangential direction, which is essentially determined by the Kähler form of the manifold. The density ρ_0 is chosen so as to minimize the potential energy, the particles being located at states of minimum available energy. V is thus a constant along directions tangential to the boundary of the droplet $\partial\mathcal{D}$, it depends only on the normal coordinate r. ω is a constant related to the derivative of the potential, in particular $\omega=\frac{1}{n}\frac{\partial V}{\partial r^2}$ (the scaling of V is such that ω is n independent).

Equation (5) shows that the effective edge dynamics involves only the time-derivative of Φ and one tangential derivative given by $\mathcal{L}\Phi$. The action is chiral in this sense. Generally the field Φ depends on the remaining tangential directions, leading to a multiplicity of modes. For a spherical droplet in even dimensions $2k$, $\partial\mathcal{D}\sim S^{2k-1}$. The mode analysis of the field Φ has been indicated in [5]. In the special case of $\mathbf{CP^1}=S^2$, the corresponding action in (5) is the well known one-dimensional chiral bosonic action describing edge excitations for $\nu=1$ two-dimensional QHE [6].

On $\mathbf{CP}^k = SU(k+1)/U(k)$, it is possible to have a nonzero background value for the $SU(k)$ gauge fields as well as the $U(1)$ fields. As mentioned in the introduction, in this paper, we shall carry out a similar large N simplification of the action for a background which has nonzero values for the $SU(k)$ and $U(1)$ subalgebras. One of the main differences in the nonabelian case is that now the lowest Landau level states belong to a representation of $SU(k+1)$ with a lowest weight state which transforms nontrivially under $SU(k)$, say, as a representation $\tilde{J}$. As a result, the classical function Φ which is the symbol corresponding to the hermitian operator $\hat{\Phi}$, is now a matrix of dimension $(dim\tilde{J} \times dim\tilde{J})$. Again, as in the abelian case, the key to extracting the large N limit of the effective action are the two properties 1 and 2 mentioned above, appropriately modified in the case of the nonabelian bakground.

Property 2 still remains true, with the difference that the symbol corresponding to the density operator is a diagonal matrix of dimension $(dim\tilde{J} \times dim\tilde{J})$ proportional to a step function, which again identifies the droplet.

Property 1 is more seriously modified. The large N limit of the operator commutator, due to the matrix nature of the symbols, will now involve a Poisson bracket-type term along with a matrix commutator term. In particular

$$[\hat{A}, \hat{B}] \to [A, B] + \frac{i}{n}\{A, B\}_{gauged} \tag{7}$$

where

$$\{A, B\}_{gauged} = (\Omega^{-1})^{\bar{i}j}(D_{\bar{i}}AD_jB - D_{\bar{i}}BD_jA) \tag{8}$$

Notice that the Poisson bracket now involves the gauge covariant derivative $D = \partial + [\mathcal{A}, \ ]$, where $\mathcal{A}$ is the $SU(k)$ gauge potential.

The relevant field in writing down the effective action is a unitary matrix G which is an element of $U(dim\tilde{J})$. As expected, since we have an $SU(k)$ background, the action has an $SU(k)$ gauge symmetry. The field space is essentially $U(dim\tilde{J})/SU(k)$. The final result turns out to be a chiral, gauged Wess-Zumino-Witten action generalized to higher dimensions.

$$\begin{aligned} \mathcal{S}(G) \ &= \ \frac{1}{4\pi^k}M^{k-1}\int_{\partial\mathcal{D}} dt \ \mathrm{tr}\left[\left(G^\dagger\dot{G} \ + \ \omega \ G^\dagger\mathcal{L}G\right)G^\dagger\mathcal{L}G\right] \\ &\quad + (-1)^{\frac{k(k-1)}{2}}\frac{i}{4\pi}\frac{M^{k-1}}{(k-1)!}\int_{\mathcal{D}} dt \ 2 \ \mathrm{tr}\left[G^\dagger\dot{G}(G^{-1}DG)^2\right] \wedge \left(\frac{i\Omega}{\pi}\right)^{k-1} \end{aligned} \tag{9}$$

Here the first term is on the boundary $\partial\mathcal{D}$ of the droplet and it is precisely the gauged, nonabelian analogue of (5). The operator $\mathcal{L}$ in (9) is the gauged version of (6),

$$\mathcal{L} = i(\xi^i D_i - \bar{\xi}^{\bar{i}} D_{\bar{i}}) \tag{10}$$

The second term in (9), written as a differential form, is a higher dimensional Wess-Zumino term; it is an integral over the rescaled droplet $\mathcal{D}$ itself, with the radial variable playing the role of the extra dimension. This term is very similar to the Wess-Zumino term appearing in Kähler-Wess-Zumino-Witten models [7], used in the context of higher dimensional conformal field theories. The effective action (9) is the main result of this paper.

3 Analysis on $\mathbf{CP}^k$ with a nonabelian background field

1. *Wavefunctions*

We begin with a discussion of the Landau level states and wavefunctions for the case when the background is nonabelian. We will follow the group theoretic analysis given in [2], [5]. Since $\mathbf{CP}^k$ is the coset space $SU(k+1)/U(k)$, the wavefunctions can be obtained as functions on $SU(k+1)$ which have a specific transformation property under the $U(k)$ subgroup. Let t_A denote the generators of $SU(k+1)$ as matrices in the fundamental representation; we normalize them by $\mathrm{Tr}(t_A t_B) = \frac{1}{2}\delta_{AB}$. The generators corresponding to the $SU(k)$ part of $U(k) \subset SU(k+1)$ will be denoted by t_a, $a = 1,\ 2, \cdots,\ k^2-1$ and the generator for $U(1)$ direction of the subgroup $U(k)$ will be denoted by t_{k^2+2k}. A basis of functions on $SU(k+1)$ is given by the Wigner $\mathcal{D}$-functions which are the matrices corresponding to the group elements in a representation J

$$\mathcal{D}^{(J)}_{L;R}(g) = \langle J, L_i|\hat{g}|J, R_i\rangle \tag{11}$$

where $L_i,\ R_i$ stand for two sets of quantum numbers specifying the states on which the generators act, for left and right $SU(k+1)$ actions on g, respectively. On an element $g \in SU(k+1)$, we can define left and right $SU(k+1)$ actions by

$$\hat{L}_A\ g = T_A\ g, \qquad\qquad \hat{R}_A\ g = g\ T_A \tag{12}$$

where T_A are the $SU(k+1)$ generators in the representation to which g belongs. Since $\mathbf{CP}^k = SU(k+1)/U(k)$, one can have, in general, a background magnetic field which is a $U(k)$ gauge field. Previously, we concentrated on the case of a $U(1)$ field, here we are interested in the case of a $U(k)$ background, where the wavefunctions transform as a specific representation, say $\tilde{J}$, of $SU(k)$, for the action of the right generators $R_a \in SU(k)$ and carry a charge corresponding to the right $U(1)$ generator R_{k^2+2k}. The $U(1)$ generators L_{k^2+2k} and R_{k^2+2k} will be referred to as the "hypercharge operators", taking over the standard terminology for $SU(3)$.

There are $2k$ right generators of $SU(k+1)$ which are not in $U(k)$; these can be separated into T_{+i}, $i = 1, 2\cdots, k$, which are of the raising type and T_{-i} which are of the lowering type. The covariant derivatives on $\mathbf{CP}^k$, in terms of their action on the wavefunctions, can be identified with these $\hat{R}_{\pm i}$ right rotations on g. This is consistent with the fact that the commutator of covariant derivatives is the magnetic field. The commutators of $\hat{R}_{+i}$ and $\hat{R}_{-i}$ are in the Lie algebra of $U(k)$; given the values of these $U(k)$ generators on the wavefunctions, we see that they correspond to constant magnetic fields. In the absence of a confining potential, the Hamiltonian $\hat{\mathcal{H}}_0$ may be reduced to the form $\sum_i \hat{R}_{+i}\hat{R}_{-i}$, apart from additive constants. The corresponding energy eigenvalues of a charged particle on $\mathbf{CP}^k$ in the presence of a nonabelian background are given by [2]

$$E = \frac{1}{2MR^2}\left[C_2^{SU(k+1)}(J) - C_2^{SU(k)}(\tilde{J}) - R^2_{k^2+2k}\right] \tag{13}$$

where M is the mass of the charged particles, R is the radius of $\mathbf{CP}^k$, C_2 is the quadratic Casimir operator for the group and representation indicated.

The states for a representation of $SU(k+1)$ can be labelled by two integers (P, Q), corresponding to a tensor of the form $\mathcal{T}^{\mu_1\cdots\mu_Q}_{\nu_1\cdots\nu_P}$ which is symmetric in all the upper indices, symmetric in all the

lower indices and traceless for any contraction between upper and lower indices; the indices $\mu,\ \nu$ take values $1, 2, \cdots, (k+1)$. In the case of a $U(1) \times SU(k)$ nonabelian background, it is convenient to label the irreducible representation of $SU(k+1)_R$ by $(p+l, q+l')$ corresponding to the tensor

$$\mathcal{T}^{\alpha_1 \ldots \alpha_q \gamma_1 \ldots \gamma_{l'}}_{\beta_1 \ldots \beta_p \delta_1 \ldots \delta_l} \equiv \mathcal{T}^{q,l'}_{p,l} \tag{14}$$

where p, q indicate $U(1)$ indices and l, l' indicate $SU(k)$ indices, namely α's and β's take the value $(k+1)$ and γ's and δ's take values $1, \cdots, k$.

The right hypercharge corresponding to (14) is

$$\sqrt{2k(k+1)} R_{k^2+2k} = -k(p-q) + l - l' = -nk \tag{15}$$

The fact that n has to be integer [2] implies that $(l-l')/k$ is an integer, thus constraining the possible $SU(k)_R$ representations $\tilde{J}$.

The single particle energy eigenvalues are now of the form

$$E = \frac{1}{2MR^2}\left[C_2^{SU(k+1)}(p+l, q+l') - C_2^{SU(k)}(l, l') - \frac{n^2 k}{2(k+1)}\right] \tag{16}$$

where

$$C_2^{SU(k+1)}(P, Q) = \frac{k}{2(k+1)}\left[P(P+k+1) + Q(Q+k+1) + \frac{2}{k}PQ\right] \tag{17}$$

Using (15), (16) and (17) we find that

$$E = \frac{1}{2MR^2}\left[q^2 + q(l + l' + \frac{l-l'}{k} + k) + n(q + l + \frac{k}{2}) + l(\frac{l-l'}{k} + 1)\right] \tag{18}$$

Since the background magnetic field and the representation (or charges) of the matter field are the given data for setting up the Landau problem, we must identify the wavefunctions for a given $SU(k)$ representation $\tilde{J} = (l, l')$ and $U(1)$ charge $-nk$. With $\tilde{J}$ and n fixed, the lowest energy eigenstates correspond to $q = 0$. The index q plays the role of the Landau level index. Notice that for $\tilde{J}$ of the form $\tilde{J} = (0, l')$, the lowest energy eigenstate is the lowest weight vector in the representation J of $SU(k+1)$ annihilated by $\hat{R}_i$. This choice of background is particularly simple and for the rest of the discussion we will consider this case. (For other backgrounds, the effective action will have the same general qualitative features of the action we find here.) So the LLL states we consider correspond to the tensor $\mathcal{T}^{l'}_p$, where $p = n - \frac{l'}{k}$ and $l' = jk,\ j = 1, 2, \cdots$. [3]

The Landau Hamiltonian $\hat{\mathcal{H}}_0$ commutes with the left action. This corresponds to the magnetic translation symmetry of the Landau problem with the left operators $\hat{L}_A$ representing magnetic translations. The degeneracy of the LLL states corresponds to the dimension of the $J = (p, l')$ representation, where

$$dim^{SU(k+1)}(P, Q) = \frac{k(P+Q+k)(k+P-1)!(k+Q-1)!}{(k!)^2 P! Q!} \tag{19}$$

[3] In deriving the LLL condition for $\mathbf{CP}^2$ in [2] we had assumed $l > l'$ (the notation (k, k') was used instead of (l, l')). With this assumption the lowest energy states within the $q = 0$ sector were of the form $(p+l, l)$. The assumption $l > l'$ is not necessary; without this assumption the lowest energy states are of the type (p, l'), with $l = 0$.

Using $p = n - j$, $l' = jk$, $j = 1, 2, \cdots$, we find

$$dimJ = dim^{SU(k+1)}(p, l') = \frac{k(n-j+jk+k)(k+n-j-1)!(k+jk-1)!}{(k!)^2(n-j)!(jk)!} \tag{20}$$

Similarly we find that

$$dim\tilde{J} = dim^{SU(k)}(0, l') = \frac{(k+jk-1)!}{(k-1)!(jk)!} \tag{21}$$

Using (20), (21) we find that

$$dimJ = dim\tilde{J}\ \frac{(n-j+jk+k)(k+n-j-1)!}{k!(n-j)!} \tag{22}$$

In the thermodynamic limit $n \sim R^2 \to \infty$ we have

$$c \equiv \frac{dimJ}{dim\tilde{J}} = \frac{N}{N'} \to \frac{n^k}{k!} \tag{23}$$

Following the analysis in [2], the completely filled LLL, $\nu = 1$, corresponds to a configuration of constant density

$$\rho_{\nu=1} \sim \frac{N}{N'\ R^{2k}} \to \frac{n^k}{k! R^{2k}} \to finite \tag{24}$$

In order to obtain a quantum Hall droplet for QHE on $\mathbf{CP}^k$ with a nonabelian background, we need a $U(1)$ charge $n \sim R^2 \to \infty$ while the dimension $N' = dim\tilde{J}$ of the $SU(k)$ representation remains finite.

The wavefunctions for the lowest Landau level are thus given by

$$\begin{aligned} \Psi^J_{m;\alpha}(g) &= \sqrt{N}\ \langle J, L|\ \hat{g}\ |J, (\tilde{J}, \alpha), -n\rangle \\ &= \sqrt{N}\ \mathcal{D}^J_{m;\alpha}(g) \end{aligned} \tag{25}$$

where N is the dimension of the representation J of $SU(k+1)$. The index $m = 1, \cdots, N$ represents the state within the $SU(k+1)$ representation J. The left generators L_A act on these as linear transformations. Further, the label α corresponds to the state $|\alpha\rangle = |J, (\tilde{J}, \alpha), -n\rangle$, $\alpha = 1, \cdots, N'$ and

$$\begin{aligned} \hat{R}_a\ \Psi^J_{m;\alpha}(g) &= (T^{\tilde{J}}_a)_{\alpha\beta}\Psi^J_{m;\beta}(g) \\ \hat{R}_{k^2+2k}\ \Psi^J_{m;\alpha}(g) &= -\frac{nk}{\sqrt{2k(k+1)}}\ \Psi^J_{m;\alpha}(g) \end{aligned} \tag{26}$$

The first of these equations shows that the wavefunctions (25) under right rotations transform as a representation of $SU(k)$, $(T^{\tilde{J}}_a)_{\alpha\beta}$ being the representation matrices for the generators of $SU(k)$ in the representation $\tilde{J}$. n is an integer characterizing the abelian part of the background field. α, β label states within the $SU(k)$ representation $\tilde{J}$ (which is itself contained in the representation J of $SU(k+1)$). The index α carried by the wavefunctions (25) is basically the gauge index. The wavefunctions are sections of a $U(k)$ bundle on $\mathbf{CP}^k$.

The wavefunctions (25) are normalized by virtue of the orthogonality theorem

$$\int d\mu(g)\ \mathcal{D}^{*J}_{m;\alpha}(g)\ \mathcal{D}^{J}_{m';\alpha'}(g)\ =\ \frac{\delta_{mm'}\delta_{\alpha\alpha'}}{N} \tag{27}$$

where $d\mu(g)$ is the Haar measure on $SU(k+1)$, normalized to unity. The orthonormality relations for wavefunctions should involve integration over the space, which is $\mathbf{CP}^k = SU(k+1)/U(k)$ rather than $SU(k+1)$ as we have done in (27). The difference has to do with the gauge transformations. This will be clear if we consider $U(k)$-invariant combinations. If $\phi^\alpha,\ \chi^\alpha$ transform under $U(k)$ as the representation $\tilde{J}$, then the combinations $\Psi^J_{m;\alpha}\phi^\alpha$ and $\Psi^J_{m;\alpha}\chi^\alpha$ are invariant and we can write

$$\begin{aligned}\int d\mu(\mathbf{CP}^k)\ (\Psi^J_{m;\beta}\chi^\beta)^*\Psi^J_{m;\alpha}\phi^\alpha\ &=\ \int d\mu(g)\ (\Psi^J_{m;\beta}\chi^\beta)^*\Psi^J_{m;\alpha}\phi^\alpha \\ &=\ \chi^\dagger\cdot\phi\end{aligned} \tag{28}$$

The integration can be extended to the full group and the orthogonality theorem (27) used by virtue of the $U(k)$-invariance of the integrand.

2. *Star products, commutators and Poisson brackets*

The simplification of the commutators in the large N-limit is achieved by use of the star product which represents the operator product in terms of classical functions corresponding to the operators. The classical function corresponding to an operator is called the symbol of the operator. To see how the symbol emerges naturally, consider the trace of a product of two operators $\hat{A}$ and $\hat{B}$. By virtue of the orthogonality theorem, we can write

$$c\sum_\alpha \int d\mu(g)\mathcal{D}^*_{m';\alpha}\ \mathcal{D}_{m;\alpha} = \delta_{m'm} \tag{29}$$

where $c = dimJ/dim\tilde{J} = N/N'$. Using this result we have

$$\mathrm{Tr}\hat{A}\hat{B} = \sum_{ml} A_{ml}B_{lm} = c\sum_{\alpha mlm'}\int d\mu(g)\ \mathcal{D}_{m;\alpha}\ A_{ml}B_{lm'}\ \mathcal{D}^*_{m';\alpha} \tag{30}$$

This shows that the appropriate definition of the symbol for $\hat{A}$ should be

$$A_{\alpha\beta}(g) = \sum_{ml}\mathcal{D}_{m;\alpha}(g)\ A_{ml}\mathcal{D}^*_{l;\beta}(g) \tag{31}$$

Notice that the symbol for an operator is now a $(N'\times N')$ matrix valued function. Under a $U(k)$ transformation, $g\to gh$, it transforms as $A_{\alpha\beta}\to h^T_{\alpha\gamma}A_{\gamma\delta}h^*_{\delta\beta}$ where h is the element of $U(k)$ in the matrix representation corresponding to $\tilde{J}$. The formula for the Wigner function, namely equation (11), shows that $\mathcal{D}_{mn}(g^T) = \mathcal{D}_{nm}(g)$. Thus the symbol for an operator $\hat{A}$ may also be written as

$$A_{\alpha\beta}(g) = \langle\alpha|\hat{g}^T\hat{A}\hat{g}^*|\beta\rangle \tag{32}$$

The symbol corresponding to the product of the operators $\hat{A}$ and $\hat{B}$ is

$$\begin{aligned}(\hat{A}\hat{B})_{\alpha\beta}(g)\ &=\ \sum_{mrm'}\mathcal{D}_{m;\alpha}(g)\ A_{mr}B_{rm'}\mathcal{D}^*_{m';\beta} \\ &=\ \sum_{mrm's}\mathcal{D}_{m;\alpha}(g)\ A_{mr}\ \delta_{rs}\ B_{sm'}\mathcal{D}^*_{m';\beta}\end{aligned} \tag{33}$$

The next step is to split the summation over the intermediate indices using the completeness relation $\mathbf{1} = \sum_A |A\rangle\langle A|$ where the summation is over all states in the $SU(k+1)$ representation. (The method outlined here for deriving the star product has already been used in [9], [5], [4].) The states can be grouped into irreducible representations of $U(k)$. The first set of terms is $\sum_\alpha |J,(\tilde{J},\alpha),-n\rangle\langle J,(\tilde{J},\alpha),-n|$. The next set of terms come from $T_{+i}|J,(\tilde{J},\alpha),-n\rangle = T_{+i}|\alpha\rangle$. (We will abbreviate the state $|J,(\tilde{J},\alpha),-n\rangle$ by $|\alpha\rangle$ when there is no likelihood of confusion.) Since T_{+i} transform as the fundamental representation of $SU(k)$, these states transform as the product of $\tilde{J}$ and the fundamental representation. If desired one can split them into irreducible components, but this is not necessary. The states $T_{+i}|\alpha\rangle$ form a complete set for the subspace with $\sqrt{2k(k+1)}\ R_{k^2+2k} = -nk+k+1$. The scalar product is given by

$$\begin{aligned}\langle\alpha|T_{-i}T_{+j}|\beta\rangle &= n\ \delta_{\alpha\beta}\ \delta_{ij} - (T_a)_{\alpha\beta}\ f^a_{ij} \\ &\equiv \mathcal{G}(\alpha i,\beta j)\end{aligned} \tag{34}$$

where we have used the commutation rule

$$[T_{-i},T_{+j}] = -\sqrt{\frac{2(k+1)}{k}}\ T_{k^2+2k}\ \delta_{ij} - f^a_{ij}\ T_a \tag{35}$$

T_a is a generator of $SU(k)$. The completeness relation, writing out the first two sets of states explicitly, becomes

$$\mathbf{1} = \sum_\alpha |\alpha\rangle\langle\alpha| \ +\ \sum_{\alpha\beta ij} T_{+i}|\alpha\rangle\ \mathcal{G}^{-1}(\alpha i,\beta j)\ \langle\beta|T_{-j}\ +\ \cdots \tag{36}$$

Putting in factors of g, this is equivalent to

$$\delta_{rs} = \sum_\gamma \mathcal{D}^*_{r,\gamma}\mathcal{D}_{s,\gamma} - \sum_{\gamma\delta ij}\mathcal{G}^{-1}(\gamma i,\delta j)\hat{R}_{-j}\mathcal{D}^*_{r,\delta}\hat{R}_{+i}\mathcal{D}_{s,\gamma} + \cdots \tag{37}$$

(The minus sign has to do with the fact that $\hat{R}_A g^\dagger = -T_A g^\dagger$.) Since

$$\mathcal{G}^{-1}(\gamma i,\delta j) = (1/n)\delta_{\gamma\delta}\delta_{ij} + \mathcal{O}(1/n^2) \tag{38}$$

we can simplify (37) as

$$\delta_{rs} = \sum_\gamma(\mathcal{D}^*_{r,\gamma}\mathcal{D}_{s,\gamma} - \frac{1}{n}\sum_{i=1}^k \hat{R}_{-i}\mathcal{D}^*_{r,\gamma}\hat{R}_{+i}\mathcal{D}_{s,\gamma}) + \mathcal{O}\left(\frac{1}{n^2}\right) \tag{39}$$

Using this in (33) we find

$$\begin{aligned}(\hat{A}\hat{B})_{\alpha\beta}(g) &= \sum_\gamma\left(A_{\alpha\gamma}(g)B_{\gamma\beta}(g) - \frac{1}{n}\sum_{i=1}^k \hat{R}_{-i}A_{\alpha\gamma}(g)\hat{R}_{+i}B_{\gamma\beta}(g)\right) + \mathcal{O}\left(\frac{1}{n^2}\right) \\ &\equiv \sum_\gamma A_{\alpha\gamma} * B_{\gamma\beta}\end{aligned} \tag{40}$$

This is the definition of the star product. The first term in the star product is thus the matrix product $\sum_\gamma A_{\alpha\gamma}B_{\gamma\beta}$; this is in contrast to the case when we have an abelian background for which

we have functions as symbols and just a product of functions as the leading term in the star product. Going back to (30) we see that it can be written as

$$\begin{aligned} \mathrm{Tr}\hat{A}\hat{B} &= c\int d\mu(g)\sum_{\alpha\beta} A_{\alpha\beta} * B_{\beta\alpha} \\ &= c\int d\mu(g)\ \mathrm{tr}(A*B) \end{aligned} \tag{41}$$

where, in the second line, the trace (denoted by lower case tr) is over the $SU(k)$ representation $\tilde{J}$ and $A,\ B$ are the symbols corresponding to $\hat{A},\ \hat{B}$.

For the symbol corresponding to the commutator of $\hat{A},\ \hat{B}$, we have

$$([\hat{A},\hat{B}])(g) = [A(g),B(g)] - \frac{1}{n}(\hat{R}_{-i}A\hat{R}_{+i}B - \hat{R}_{-i}B\hat{R}_{+i}A) + \mathcal{O}\left(\frac{1}{n^2}\right) \tag{42}$$

The first term on the right hand side of (42) is a matrix commutator, given the fact that the symbols $A(g),\ B(g)$ are matrices. This term was absent in the case of the abelian background. Further in the abelian case, the second term on the right hand side of (42) was found to be proportional to the Poisson bracket $\{A,B\}$ [5]. In the case of nonabelian background this relation is more involved due to the presence of the nonabelian gauge fields, as we shall now discuss.

A convenient parametrization of the local complex coordinates of $\mathbf{CP}^k$ is given by

$$g_{i,k+1} = \frac{\xi_i}{\sqrt{1+\bar{\xi}\cdot\xi}},\quad i=1,\cdots,k \qquad\qquad g_{k+1,k+1} = \frac{1}{\sqrt{1+\bar{\xi}\cdot\xi}} \tag{43}$$

where g is an $SU(k+1)$ group element (fundamental representation). Since g and gh, $h\in U(k)$, are identified and correspond to the same point on $\mathbf{CP}^k = SU(k+1)/U(k)$, we can use the freedom of h transformations to write g as a function of the coset coordinates $\xi^i,\ \bar{\xi}^i$ alone. We can then write

$$g^{-1}dg = -it_{+i}E^i_j d\xi^j - it_{-i}E^{\bar{i}}_{\bar{j}}d\bar{\xi}^j - it_a E^a_j d\xi^j - it_a E^a_{\bar{j}} d\bar{\xi}^j \tag{44}$$

where a takes values $1,\cdots,k^2$ and k^2+2k, and $i=1,\cdots,k$. The generators t_A of the $SU(k+1)$ algebra are divided into two sets, with t_a belonging to the $U(k)$ algebra and the lowering and raising coset operators $t_{\pm i}$, such that $\mathrm{tr}(t_a t_{\pm i})=0$. (Notice that the components $E^i_{\bar{j}}$ and $E^{\bar{i}}_j$ are zero; this follows from (43) and is consistent with the Kähler nature of $\mathbf{CP}^k$.)

The Cartan-Killing metric on $\mathbf{CP}^k$ is given by

$$ds^2 = E^i_j\ E^{\bar{i}}_{\bar{l}}\ d\xi^j\ d\bar{\xi}^l \tag{45}$$

which shows that the E's are frame fields on the coset. The Kähler two-form on $\mathbf{CP}^k$ is likewise written as

$$\begin{aligned} \Omega &= -i\sqrt{\frac{2k}{k+1}}\mathrm{tr}\left(t_{k^2+2k}\ g^{-1}dg\wedge g^{-1}dg\right) \\ &= i\ E^i_j\ E^{\bar{i}}_{\bar{l}}\ d\xi^j\wedge d\bar{\xi}^l \ \equiv\ \Omega_{j\bar{l}}\ d\xi^j\wedge d\bar{\xi}^l \end{aligned} \tag{46}$$

From (44) we can write

$$\begin{aligned}(E^{-1})_i^j \frac{\partial g}{\partial \xi^j} &= g\left[-it_{+i} - it_a E_j^a (E^{-1})_i^j\right] \\ (E^{-1})_{\bar{i}}^{\bar{j}} \frac{\partial g}{\partial \bar{\xi}^j} &= g\left[-it_{-i} - it_a E_{\bar{j}}^a (E^{-1})_{\bar{i}}^{\bar{j}}\right]\end{aligned} \tag{47}$$

We now define covariant derivatives on g by

$$\begin{aligned} D_i g = \partial_i g - g\mathcal{A}_i, \qquad & D_{\bar{i}} g = \partial_{\bar{i}} g - g\mathcal{A}_{\bar{i}} \\ \mathcal{A}_i = -it_a E_i^a, \qquad & \mathcal{A}_{\bar{i}} = -it_a E_{\bar{i}}^a \end{aligned} \tag{48}$$

Then (47) becomes

$$(E^{-1})_i^j D_j g = -ig\ t_{+i}, \qquad (E^{-1})_{\bar{i}}^{\bar{j}} D_{\bar{j}} g = -ig\ t_{-i} \tag{49}$$

Thus the right translation operators $\hat{R}_{\pm i}$ defined by $\hat{R}_{\pm i} g = g t_{\pm i}$ can be identified as

$$\hat{R}_{+i} = i(E^{-1})_i^j D_j \qquad \hat{R}_{-i} = i(E^{-1})_{\bar{i}}^{\bar{j}} D_{\bar{j}} \tag{50}$$

The gauge field $\mathcal{A} = -it_a E^a$ is the potential corresponding to the background magnetic field. It undergoes gauge transformations under right $U(k)$ rotations.

$$\begin{aligned} g^{-1}dg &= E + \mathcal{A} \\ (gh)^{-1}d(gh) &= h^{-1}Eh + h^{-1}\mathcal{A}h + h^{-1}dh \end{aligned} \tag{51}$$

where E and $\mathcal{A}$ are one-forms. Since $h^{-1}dh \in U(k)$, we find that the net effect is the gauge transformation

$$\mathcal{A} \to h^{-1}\mathcal{A}h + h^{-1}dh \qquad E \to h^{-1}Eh \tag{52}$$

The symbol of an operator $\hat{A}$ has been defined as $A_{\alpha\beta}(g) = \langle\alpha|\hat{g}^T \hat{A} \hat{g}^*|\beta\rangle$ in (32); thus the action of the right operator $\hat{R}_{-i}$ on a symbol is

$$\begin{aligned} \hat{R}_{+i} A_{\alpha\beta} &= i(E^{-1})_i^j (D_j A)_{\alpha\beta} \\ D_j A &= \partial_j A + [\mathcal{A}_j,\ A] \end{aligned} \tag{53}$$

where $\mathcal{A}_i = -iT_a E_i^a$ and T_a are $U(k)$ generators in the $\tilde{J}$-representation. Notice that the $U(1)$ part of the gauge field does not contribute in (53). (A similar formula holds for $\hat{R}_{-i}$.) Using (46) and (53) we can now write

$$\begin{aligned} \hat{R}_{-i}A\hat{R}_{+i}B - \hat{R}_{-i}B\hat{R}_{+i}A &= -(E^{-1})_{\bar{i}}^{\bar{j}}(E^{-1})_i^m (D_{\bar{j}}AD_mB - D_{\bar{j}}BD_mA) \\ &= -i(\Omega^{-1})^{\bar{j}m}(D_{\bar{j}}AD_mB - D_{\bar{j}}BD_mA) \end{aligned} \tag{54}$$

The final form for the symbol corresponding to the commutator of $\hat{A},\ \hat{B}$ in (42) is now written as

$$([\hat{A}, \hat{B}])(g) = [A(g), B(g)] + \frac{i}{n}(\Omega^{-1})^{\bar{j}m}(D_{\bar{j}}AD_mB - D_{\bar{j}}BD_mA) + \mathcal{O}\left(\frac{1}{n^2}\right) \tag{55}$$

In terms of the complex coordinates $\xi,\ \bar{\xi}$ we have

$$(\Omega^{-1})^{\bar{j}m} = -i(1+\bar{\xi}\cdot\xi)(\delta^{jm}+\bar{\xi}^j\xi^m) \tag{56}$$

In the case of an abelian background the covariant derivatives reduce to the regular derivatives and we obtain the result in [5].

3. *Simplification of the kinetic energy term*

We will first consider the $N\to\infty$ limit of the kinetic term of the action (1). We start by writing $\hat{U}=\exp(i\hat{\Phi})$. The symbol corresponding to $\hat{\Phi}$ is now a $(N'\times N')$-matrix $\Phi_{\alpha\beta}$. In expanding $\hat{U}$ we encounter products of $\hat{\Phi}$'s, which have to be simplified by use of star products. Consider for example $\hat{\Phi}^2$. The symbol corresponding to this may be written as

$$\Phi * \Phi = \Phi\ \Phi - \frac{1}{n}\hat{R}_{-i}\Phi\ \hat{R}_{+i}\Phi + \cdots \tag{57}$$

where on the right hand side we have the matrix product of Φ's. (From now on we assume that expressions with repeated indices involve a summation.) Let e_A form a basis for $(N'\times N')$ matrices, so that we can write $\Phi = e_A\Phi^A$. The star product of the Φ's in (57) can then be written as

$$\begin{aligned}\Phi * \Phi &= e_A e_B\left(\Phi^A - \frac{1}{n}\hat{R}_{-i}\Phi^A\hat{R}_{+i}\Phi^C\frac{\partial}{\partial\Phi^C}\right)\ \left(\Phi^B - \frac{1}{n}\hat{R}_{-j}\Phi^B\hat{R}_{+j}\Phi^D\frac{\partial}{\partial\Phi^D}\right)\cdot 1\\ &= \tilde{\Phi}\ \tilde{\Phi}\end{aligned} \tag{58}$$

where

$$\begin{aligned}\tilde{\Phi} &= e_A\left(\Phi^A - \frac{1}{n}\hat{R}_{-i}\Phi^A\hat{R}_{+i}\Phi^C\frac{\partial}{\partial\Phi^C}\right)\\ &= e_A\left(\Phi^A + \xi^{AC}\partial_C\right)\\ \xi^{AC} &= -\frac{1}{n}\hat{R}_{-i}\Phi^A\hat{R}_{+i}\Phi^C\end{aligned} \tag{59}$$

where $\partial_C = \partial/\partial\Phi^C$. It is easily checked that a similar formula holds for all higher powers of $\hat{\Phi}$'s, so that the simplification, at least to the leading $1/n$ order, can be carried out in a closed form. (This way of simplifying operator products is very close to the standard operator product expansions in field theory. Although not needed for our purpose, the expansion can be carried out to any order in $1/n$ by a similar formula.) We now write the symbol corresponding to $\hat{U}$ as

$$\begin{aligned}U &= \exp(i\tilde{\Phi}) = \exp(ie_A(\Phi^A+\xi^{AC}\partial_C))\\ &= e^{i\Phi}\ +\ \int_0^1 d\alpha\ e^{i\Phi(1-\alpha)}ie_A\xi^{AC}\partial_C e^{i\Phi\alpha}\ +\cdots\\ &= G\ -\ \frac{i}{n}G\ F+\cdots\end{aligned} \tag{60}$$

where

$$F = \int_0^1 d\alpha\ e^{-i\alpha\Phi}\ \hat{R}_{-i}\Phi\ \hat{R}_{+i}(e^{i\alpha\Phi}) \tag{61}$$

and G is the unitary $(N' \times N')$-matrix $e^{i\Phi}$. In a similar way, we find

$$\begin{aligned} U^\dagger &= G^\dagger + \frac{i}{n} F^\dagger G^\dagger + \cdots \\ F^\dagger &= \int_0^1 d\alpha\ \hat{R}_{-i}(e^{-i\alpha\Phi})\ \hat{R}_{+i}\Phi\ e^{i\alpha\Phi} \end{aligned} \tag{62}$$

Using these results we find

$$U^\dagger * \dot{U} = G^\dagger \dot{G} - \frac{1}{n}\hat{R}_{-i}G^\dagger\ \hat{R}_{+i}\dot{G} - \frac{i}{n}\dot{F} - \frac{i}{n}G^\dagger \dot{G} F + \frac{i}{n}F^\dagger G^\dagger \dot{G} + \cdots \tag{63}$$

where the overdot denotes differentiation with respect to time t. The $\dot{F}$-term will give a total time-derivative upon taking the trace (or integral) with ρ_0, so we shall drop it in what follows. The term $\hat{R}_{-i}G^\dagger\ \hat{R}_{+i}\dot{G}$ can be simplified as

$$\begin{aligned} \hat{R}_{-i}G^\dagger\ \hat{R}_{+i}\dot{G} &= \hat{R}_{+i}\left(\hat{R}_{-i}G^\dagger\ \dot{G}\right) - \hat{R}_{+i}\hat{R}_{-i}G^\dagger\ \dot{G} \\ &= \frac{\partial}{\partial t}\left(\hat{R}_{-i}G^\dagger\ \hat{R}_{+i}G\right) - \hat{R}_{-i}\dot{G}^\dagger \hat{R}_{+i}G \\ &= \frac{\partial}{\partial t}\left(\hat{R}_{-i}G^\dagger\ \hat{R}_{+i}G\right) - \hat{R}_{-i}\left(\dot{G}^\dagger \hat{R}_{+i}G\right) + \dot{G}^\dagger \hat{R}_{-i}\hat{R}_{+i}G \\ &= \frac{1}{2}\frac{\partial}{\partial t}\left(\hat{R}_{-i}G^\dagger\ \hat{R}_{+i}G\right) + \frac{1}{2}\hat{R}_{+i}(\hat{R}_{-i}G^\dagger \dot{G}) - \frac{1}{2}\hat{R}_{-i}(\dot{G}^\dagger \hat{R}_{+i}G) \\ &\qquad -\frac{1}{2}\hat{R}_{+i}\hat{R}_{-i}G^\dagger\ \dot{G} + \frac{1}{2}\dot{G}^\dagger \hat{R}_{-i}\hat{R}_{+i}G \end{aligned} \tag{64}$$

Derivatives on $G^\dagger$ can be written in terms of derivatives on G; carrying this out for the relevant terms and rearranging, the expression given above can be written as

$$\begin{aligned} \hat{R}_{-i}G^\dagger\ \hat{R}_{+i}\dot{G} &= \frac{1}{2}\frac{\partial}{\partial t}\left(\hat{R}_{-i}G^\dagger\ \hat{R}_{+i}G\right) + \frac{1}{2}\hat{R}_{+i}(\hat{R}_{-i}G^\dagger \dot{G}) - \frac{1}{2}\hat{R}_{-i}(\dot{G}^\dagger \hat{R}_{+i}G) \\ &\qquad +\frac{1}{4}\left[G^\dagger \dot{G},\ (\hat{R}_{+i}\hat{R}_{-i}G^\dagger)G\right] - \frac{1}{4}\left[G^\dagger \dot{G},\ G^\dagger(\hat{R}_{+i}\hat{R}_{-i}G)\right] \\ &\qquad -\frac{1}{4}\left[G^\dagger \dot{G},\ G^\dagger \hat{R}_{-i}G\ G^\dagger \hat{R}_{+i}G + G^\dagger \hat{R}_{+i}G\ G^\dagger \hat{R}_{-i}G\right]_+ \end{aligned} \tag{65}$$

where the square brackets with the subscript + indicate the anticommutator of the terms involved.

The expression for F can be written as

$$\begin{aligned} F &= \int_0^1 d\alpha e^{-i\alpha\Phi}\hat{R}_{-i}\Phi\ \hat{R}_{+i}(e^{i\alpha\Phi}) \\ &= \int_0^1 d\alpha\ \hat{R}_{+i}\left(e^{-i\alpha\Phi}\hat{R}_{-i}\Phi\ e^{i\alpha\Phi}\right) - \int_0^1 d\alpha\ \hat{R}_{+i}\left(e^{-i\alpha\Phi}\hat{R}_{-i}\Phi\right)\ e^{i\alpha\Phi} \\ &= -i\hat{R}_{+i}(G^\dagger \hat{R}_{-i}G) - \int_0^1 d\alpha\ \hat{R}_{+i}\left(e^{-i\alpha\Phi}\hat{R}_{-i}\Phi\right)\ e^{i\alpha\Phi} \end{aligned} \tag{66}$$

In a similar way, $F^\dagger$ can be written as

$$F^\dagger = i\hat{R}_{+i}(\hat{R}_{-i}G^\dagger)\ G - \int_0^1 d\alpha\ \hat{R}_{+i}\left(e^{-i\alpha\Phi}\hat{R}_{-i}\Phi\right)e^{i\alpha\Phi} \tag{67}$$

This shows also that

$$F - F^\dagger = -iG^\dagger \hat{R}_{-i} G \; G^\dagger \hat{R}_{+i} G \tag{68}$$

We now rewrite the last set of terms in (63) as

$$\begin{aligned} iG^\dagger \dot{G} F - iF^\dagger G^\dagger \dot{G} &= \frac{i}{2}\left[G^\dagger \dot{G},\; F + F^\dagger\right] + \frac{i}{2}\left[G^\dagger \dot{G},\; F - F^\dagger\right]_+ \\ &= \frac{i}{2}\left[G^\dagger \dot{G},\; F + F^\dagger\right] + \frac{1}{2}\left[G^\dagger \dot{G},\; G^\dagger \hat{R}_{-i} G \; G^\dagger \hat{R}_{+i} G\right]_+ \end{aligned} \tag{69}$$

Using (65) and (69), we can finally write the kinetic term in the action as

$$K.E. = ic \int dt d\mu \; \mathrm{tr}\rho_0 G^\dagger \dot{G} + ic \int dt d\mu \mathrm{tr}\rho_0 \left[G^\dagger \dot{G},\; Q\right] \;+\; c\mathcal{S}_1 \tag{70}$$

where the antihermitian matrix Q is given by

$$Q = -\frac{i}{2n}(F + F^\dagger) - \frac{1}{4n}(\hat{R}_{+i}\hat{R}_{-i}G^\dagger)G + \frac{1}{4n} G^\dagger \hat{R}_{+i}\hat{R}_{-i} G \tag{71}$$

We have used the fact that $\sum_i [\hat{R}_{-i},\; \hat{R}_{+i}]G = 0$, since $\sum_i [\hat{R}_{-i},\; \hat{R}_{+i}]$ is proportional to the (right) hypercharge. The expression $\mathcal{S}_1$ is given by

$$\mathcal{S}_1 = \frac{i}{2n} \int dt d\mu \; \mathrm{tr}\rho_0 \left\{ \hat{R}_{-i}(\hat{R}_{+i} G^\dagger \; \dot{G}) - \hat{R}_{+i}(\dot{G}^\dagger \hat{R}_{-i} G) + \frac{1}{2}\left[G^\dagger \dot{G},\; [G^\dagger \hat{R}_{+i} G, G^\dagger \hat{R}_{-i} G]\right]_+ \right\} \tag{72}$$

(As mentioned before we have dropped the total time-derivative term.)

The first term in the kinetic energy term (70) is a bulk term whose interpretation we shall discuss shortly. The second term is also a bulk term. It can be absorbed into the first term by a field redefinition. Define $\tilde{G} = G\exp(Q)$; we then find

$$i \int \mathrm{tr}\rho_0 \tilde{G}^\dagger \dot{\tilde{G}} = i \int \mathrm{tr}\rho_0 G^\dagger \dot{G} + i \int \mathrm{tr}\rho_0 \left[G^\dagger \dot{G},\; Q\right] \tag{73}$$

Thus the second term is absorbed into the first if we use $\tilde{G}$ in place of G. Since the change is of order $1/n$, we can do this in $\mathcal{S}_1$ as well; it has the same form with G replaced by $\tilde{G}$, corrections being of order $1/n^2$. The final answer for the kinetic term is then

$$K.E. = ic \int dt \; d\mu \; \mathrm{tr}\rho_0 G^\dagger \dot{G} \;+\; c\mathcal{S}_1(G) \tag{74}$$

where we have dropped the tildes on G, since we will be using this redefined G from now on.

We now make some observations about the scaling behaviour for various terms with n, which will be useful in the ensuing discussion. The local coordinates ξ^i introduced in (43) are dimensionless and we identify $R\xi^i$ as the coordinates of the manifold, where R is the radius of $\mathbf{CP}^k$. The $\mathbf{CP}^k$ measure of integration $d\mu$ is written in terms of the coordinates ξ as [5]

$$\begin{aligned} d\mu &= \frac{k!}{\pi^k} \frac{d^k\xi d^k\bar{\xi}}{(1 + \bar{\xi}\cdot\xi)^{k+1}} \\ &= i^k \frac{k! \det\Omega}{\pi^k} \; d^k\xi \; d^k\bar{\xi} \end{aligned} \tag{75}$$

Since $R \sim \sqrt{n}$, the large n limit is taken keeping $\sqrt{n}\ \xi$ finite. So a term like $c\int d\mu$ will go like $cn^{-k} \sim 1$ as $n \to \infty$ since $c \sim n^k$ as shown in (23). Such terms will be the bulk terms in the action, for example, the term $\int \mathrm{tr}(\rho_0 G^\dagger \dot{G})$. For such terms, it is sufficient to retain the leading term of order one in ρ_0, the subleading terms vanish as $n \to \infty$. One can also have terms with two derivatives $\hat{R}_{+i}$ and $\hat{R}_{-i}$, which arise with a power of $1/n$ in the action. (These are typically boundary terms for the droplet as we shall see later.) The rescaling of these derivative terms will produce an additional power of n, so that a finite result is again obtained with the leading term in ρ_0. Thus for all terms in the action, it is sufficient to consider the leading term in the density.

The next step will be the simplification of the potential energy term. But before we do this, we shall make a digression on how to evaluate the symbol $(\rho_0)_{\alpha\beta}$ corresponding to the density operator; this will help to simplify some of the calculations later.

4. *A formula for the large $n-$limit*

We now consider the symbol for the product $\hat{T}_B\hat{A}$ where $\hat{T}_B$ are the generators of $SU(k+1)$, viewed as linear operators on the states in the representation J. Using the formula (32), it can be simplified as follows.

$$\begin{aligned}
(\hat{T}_B\hat{A})_{\alpha\beta} &= \langle\alpha|\ \hat{g}^T\ \hat{T}_B\ \hat{A}\ \hat{g}^*\ |\beta\rangle \\
&= S_{BC}\ \langle\alpha|\ \hat{T}_C\ \hat{g}^T\ \hat{A}\ \hat{g}^*\ |\beta\rangle \\
&= S_{Ba}(T_a)_{\alpha\gamma}\ \langle\gamma|\ \hat{g}^T\ \hat{A}\ \hat{g}^*\ |\beta\rangle + S_{B+i}\ \langle\alpha|\ \hat{T}_{-i}\ \hat{g}^T\ \hat{A}\ \hat{g}^*\ |\beta\rangle \\
&\qquad + S_{B\ k^2+2k}\ \langle\alpha|\ \hat{T}_{k^2+2k}\ \hat{g}^T\ \hat{A}\ \hat{g}^*\ |\beta\rangle \\
&= \mathcal{L}_{B\alpha\gamma}\ \langle\gamma|\ \hat{g}^T\ \hat{A}\ \hat{g}^*\ |\beta\rangle \\
&= \mathcal{L}_{B\alpha\gamma}\ A(g)_{\gamma\beta}
\end{aligned} \tag{76}$$

where we have used $\hat{g}^T\hat{T}_B\hat{g}^* = S_{BC}\hat{T}_C$, $S_{BC} = 2\mathrm{Tr}(g^T t_B g^* t_C)$. (Here t_B, t_C and the trace are in the fundamental representation of $SU(k+1)$.) $\mathcal{L}_B$ is defined as

$$\mathcal{L}_{B\alpha\gamma} = -\delta_{\alpha\gamma}\frac{nk}{\sqrt{2k(k+1)}}S_{B\ k^2+2k} \ + \ \delta_{\alpha\gamma}S_{B+i}\hat{\tilde{R}}_{-i} + S_{Ba}(T_a)_{\alpha\gamma} \tag{77}$$

and $\hat{\tilde{R}}_{-i}$ is a differential operator defined by $\hat{\tilde{R}}_{-i}g^T = T_{-i}g^T$. (This can be related to $\hat{R}_{-i}$ but it is immaterial here.) By choosing $\hat{A}$ as a product of $\hat{T}$'s, we can extend the calculation of the symbol for any product of $\hat{T}$'s using equation (76). We find

$$(\hat{T}_A\hat{T}_B\cdots\hat{T}_M)_{\alpha\beta} = \mathcal{L}_{A\alpha\gamma_1}\mathcal{L}_{B\gamma_1\gamma_2}\cdots\mathcal{L}_{M\gamma_r\beta}\cdot 1 \tag{78}$$

When n becomes very large, the term that dominates in $\mathcal{L}_A$ is $S_{A\ k^2+2k}$. These correspond to the coordinates of the $\mathbf{CP}^k$ space; $S_{A\ k^2+2k}$ obey algebraic constraints which ensure that we are describing $\mathbf{CP}^k$ embedded in $\mathbf{R}^{k^2+2k}$. (The above relations also apply when the background is abelian so that the lowest weight state is an $SU(k)$ singlet. In this case the term with $(T_a)_{\alpha\beta}$ is absent in $\mathcal{L}_{A\alpha\beta}$.)

5. *Evaluating the density*

The density is a function of the matrices T_A in the representation J of $SU(k+1)$. The method of evaluation of the symbol given above, eqs. (77)-(78), shows that its large n-limit corresponds to replacing the $\hat{T}_A$'s by $-\ (nk/\sqrt{2k(k+1)}\ )S_{A\ k^2+2k}\ \delta_{\alpha\beta}$. In particular, the leading term of the density will be a singlet under right $SU(k)$. There will be nondiagonal subleading terms, which are down by a power of n, which involve one power of the matrix $(T_a)_{\alpha\beta}$.

A more explicit analysis of the density will require more details about the construction of the states. We found in section 3.1 that the LLL states are essentially states of the J representation of $SU(k+1)$ whose lowest weight state is the $\tilde{J}$ representation of $SU(k)$. The lowest weight state corresponds to the tensor $\mathcal{T}^{i_1 i_2 \cdots i_{l'}}_{(k+1)(k+1)\cdots(k+1)} \equiv \mathcal{T}^{l'}_p$, where i's take values $1, 2, \cdots, k$ and all the lower p indices are of the $U(1)$ type and they are set to $k+1$. This transforms as the symmetric rank l' representation of $SU(k)$ and has hypercharge given by (26) with $nk = pk + l'$. This representation may also be written as the product $\mathcal{T}^{i_1 i_2 \cdots i_{l'}} \times \mathcal{T}_{(k+1)(k+1)\ldots(k+1)}$. Going back to the notation for states we used earlier, this can be written as

$$\begin{aligned} |\alpha\rangle \equiv |i_1, i_2, ..., i_{l'}; (k+1), (k+1), ..., (k+1)\rangle &= |i_1, i_2, ..., i_{l'}\rangle \times |k+1\rangle \times |k+1\rangle \cdots \times |k+1\rangle \\ &= |\tilde{J}, \alpha\rangle \times |k+1\rangle \times |k+1\rangle \cdots \times |k+1\rangle \end{aligned} \tag{79}$$

At the next level, we have the states $T_{+i}\ \mathcal{T}^{i_1 i_2 \ldots i_{l'}}_{(k+1)(k+1)\ldots(k+1)}$ which are states of higher hypercharge in the representation J. We can obtain all the states of the representation J by successive application of T_{+i}'s. The states are naturally grouped into multiplets of $SU(k)$ with increasing values of hypercharge.

We will now consider the simplification of the density when all the $SU(k)$ multiplets upto a fixed hypercharge are completely filled, starting from the lowest, to form the droplet. This can be achieved by choosing the potential to be linear on the hypercharge operator. In particular we choose it to be of the form

$$\hat{V} = \sqrt{\frac{2k}{k+1}}\ \omega\ \left(\hat{T}_{k^2+2k} + \frac{nk}{\sqrt{2k(k+1)}} \right) \tag{80}$$

where ω is a constant. (The potential does not have to be exactly of this form; any potential with the same qualitative features will do. We use this specific case to carry out calculations explicitly.) The particular expression (80) is such that

$$\langle s|\hat{V}|s\rangle \tag{81}$$

where $|s\rangle$ denotes an $SU(k)$ multiplet of hypercharge $-nk+sk+s$, namely $\sqrt{2k(k+1)}\hat{T}_{k^2+2k}|s\rangle = (-nk+sk+s)|s\rangle$.

When a number of $SU(k)$ multiplets up to a certain value of hypercharge is filled, the density operator is

$$\hat{\rho}_0 = \sum_{s=0}^{M} |s\rangle\langle s| \tag{82}$$

The corresponding symbol for the density is

$$(\rho_0)_{\alpha\beta} = \sum_{s=0}^{M} \mathcal{D}_{s;\alpha}(g)\mathcal{D}^*_{s;\beta}(g) \tag{83}$$

The first set of terms in $(\rho_0)_{\alpha\beta}$ in (83), the term for $s=0$, is given by $\sum_\gamma \langle\alpha|\hat{g}^T|\gamma\rangle\langle\gamma|\hat{g}^*|\beta\rangle$. Using the fact that the lowest weight state is a product as in (79), we get

$$\sum_\gamma \langle\alpha|\hat{g}^T|\gamma\rangle\langle\gamma|\hat{g}^*|\beta\rangle = (g_{k+1,k+1}\ g^*_{k+1,k+1})^p \langle i_1...i_{l'}|\hat{g}^T|k_1...k_{l'}\rangle\langle k_1...k_{l'}|\hat{g}^*|j_1...j_{l'}\rangle \tag{84}$$

where g is a $(k+1)\times(k+1)$ matrix as in (43). Further since the state $|j_1...j_{l'}\rangle$ is itself a product of states $|j_1\rangle|j_2\rangle...|j_{l'}\rangle$ with suitable symmetrizations, the term $\langle i_1...i_{l'}|\hat{g}^T|k_1...k_{l'}\rangle\langle k_1...k_{l'}|\hat{g}^*|j_1...j_{l'}\rangle$ is a product of terms like $g^T_{i_1k_1}g^*_{k_1j_1}$. Adding and subtracting the term $g^T_{i_1(k+1)}g^*_{(k+1)j_1}$ and using unitarity of the g's we get

$$\begin{aligned} g^T_{i_1k_1}\ g^*_{k_1j_1} &= \delta_{i_1j_1} - g^T_{i_1(k+1)}g^*_{(k+1)j_1} \\ &= \delta_{i_1j_1}\ + \mathcal{O}(1/n) \end{aligned} \tag{85}$$

The second line in the above equation results from the fact that $g_{i(k+1)} \sim \xi_i$, eq. (43), and the scaling properties given after (74). Further since l' is finite and $p \to n$ for large n, we find that (84) can be written as

$$(\rho)^{(s=0)}_{\alpha\beta} = \sum_\gamma \langle\alpha|\hat{g}^T|\gamma\rangle\langle\gamma|\hat{g}^*|\beta\rangle = (g_{k+1,k+1}\ g^*_{k+1,k+1})^p\ \ \delta_{\alpha\beta} \tag{86}$$

$$\sim \frac{\delta_{\alpha\beta}}{(1+\bar{\xi}\cdot\xi)^n} \tag{87}$$

Consider now the $s=1$ term in the expression for $(\rho_0)_{\alpha\beta}$ in (83). It is given by

$$\rho^{(s=1)}_{\alpha\beta} = \sum_{\gamma\delta i'j'} \langle\alpha|\hat{g}^T\ \hat{T}_{+i'}|\gamma\rangle\ \mathcal{G}^{-1}(\gamma i',\delta j')\ \langle\delta|\hat{T}_{-j'}\ \hat{g}^*|\beta\rangle \tag{88}$$

When $T_{+i'}$ acts on $|\gamma\rangle$, it can either act on the $U(1)$ indices of the state converting them to $SU(k)$ indices or on the $SU(k)$ indices. The first operation produces a factor p since there are p $U(1)$ indices, while the second one produces a finite factor. For large n the first set is dominant since $p\sim n$, so we can write

$$\begin{aligned} \langle\alpha|\hat{g}^T\ T_{+i'}|\gamma\rangle &= p(g_{(k+1)(k+1)})^{p-1}g_{(k+1)i'}\langle i_1...i_s|\hat{g}^T|\gamma\rangle\ + \cdots \\ &\sim n(g_{(k+1)(k+1)})^n\bar{\xi}_{i'}\langle i_1...i_s|\hat{g}^T|\gamma\rangle\ + \cdots \end{aligned} \tag{89}$$

This result then gives for large n

$$\begin{aligned} \rho^{(s=1)}_{\alpha\beta} &= n^2(g_{k+1,k+1}\ g^*_{k+1,k+1})^n\bar{\xi}_{i'}\ \mathcal{G}^{-1}(\gamma i',\delta j')\ \xi_{j'}\langle\alpha|\hat{g}^T|\gamma\rangle\langle\gamma|\hat{g}^*|\beta\rangle\ + \cdots \\ &= n\bar{\xi}\cdot\xi\ \frac{\delta_{\alpha\beta}}{(1+\bar{\xi}\cdot\xi)^n} + \cdots \end{aligned} \tag{90}$$

where (38) was used to simplify $\mathcal{G}^{-1}$ in (90). Notice that since $n\bar{\xi}\cdot\xi$ is fixed as $n\to\infty$, the first term is finite and the others are negligible at large n. Working out in a similar fashion the large n contribution of the higher s terms in $(\rho_0)_{\alpha\beta}$ we find

$$\begin{aligned} (\rho_0)_{\alpha\beta} &= [1+n\bar{\xi}\cdot\xi+\cdots]\ \frac{\delta_{\alpha\beta}}{(1+\bar{\xi}\cdot\xi)^n} \\ &= \sum_{s=0}^{M}\left[\frac{n!}{s!(n-s)!}\right]\frac{(\bar{\xi}\cdot\xi)^s}{(1+\bar{\xi}\cdot\xi)^n}\ \delta_{\alpha\beta} \end{aligned} \tag{91}$$

Thus the density is proportional to $\delta_{\alpha\beta}$ and the proportionality factor is exactly the scalar density ρ_0 we obtained for the abelian background [5]. We thus get for the large n limit

$$\begin{aligned}(\rho_0)_{\alpha\beta} &= \rho_0\ \delta_{\alpha\beta} \\ &\approx \Theta\left(1 - n\frac{\xi\cdot\bar{\xi}}{M}\right)\ \delta_{\alpha\beta} \end{aligned} \tag{92}$$

The proof that the density for the abelian background is a step function for $N \to \infty$ and large M was given in detail in [5]. The density in (92) corresponds to a nonabelian droplet configuration with boundary defined by $n\bar{\xi}\cdot\xi = M$. The radius of the droplet is proportional to $\sqrt{M}$.

6. *Simplification of the potential energy term*

The large n limit of the potential energy term can now be derived in a fairly straightforward manner. First we calculate the symbol corresponding to $\hat{V}$ as defined in (80).

$$V_{\alpha\beta} = \langle\alpha|\hat{g}^T\hat{V}\hat{g}^*|\beta\rangle \tag{93}$$

$$= \omega\sqrt{\frac{2k}{k+1}}\left[\langle\alpha|\hat{g}^T\hat{T}_{k^2+2k}\hat{g}^*|\beta\rangle + \frac{nk\delta_{\alpha\beta}}{\sqrt{2k(k+1)}}\right] \tag{94}$$

Using (76), (77) we find

$$\langle\alpha|\hat{g}^T\hat{T}_{k^2+2k}\ \hat{g}^*|\beta\rangle = -\delta_{\alpha\beta}\frac{nk}{\sqrt{2k(k+1)}}S_{k^2+2k,\ k^2+2k}\ + S_{k^2+2k,a}(T_a)_{\alpha\beta} \tag{95}$$

In the large n limit the first term in the above equation dominates, so combining (94) and (95) we find that the symbol for the potential is

$$V_{\alpha\beta}(g) = V(r)\ \delta_{\alpha\beta} \tag{96}$$

where $V(r)$ is the symbol for the potential in the case of the abelian background [5]. This depends only on the radial coordinate $r^2 = \bar{\xi}\cdot\xi$ and it was found in [5] to be

$$V(r) = \omega n\frac{\bar{\xi}\cdot\xi}{1+\bar{\xi}\cdot\xi} \tag{97}$$

The potential energy term can be written as

$$P.E. = \mathrm{Tr}\hat{\rho}_0\hat{U}^\dagger\hat{V}\hat{U} = \sum_{l=0}^{\infty}\frac{(-i)^l}{l!}\mathrm{Tr}\hat{\rho}_0[\hat{\Phi},[\hat{\Phi},[...,[\hat{\Phi},\hat{V}]...] \tag{98}$$

$$= \sum_{l=0}^{\infty}\frac{(-i)^l}{l!}\mathrm{Tr}[\hat{\rho}_0,\hat{\Phi}][\hat{\Phi},[\hat{\Phi},[...,[\hat{\Phi},\hat{V}]...] \tag{99}$$

where on the right hand side of (99) there are l $\hat{\Phi}$'s involved. The first term in this sum, ($l = 0$) is a constant field-independent term. The term with $l = 1$ is $\mathrm{Tr}[\hat{V},\hat{\rho}_0]\hat{\Phi}$. This is zero by the fact that ρ_0 is at the minimum of the potential.

Given the result (92) for the density, the matrix commutator term $[\rho_0,\ \Phi] = 0$, so that

$$\begin{aligned}([\hat{\rho}_0, \hat{\Phi}]) &= -\frac{1}{n}(\hat{R}_{-i}\rho_0\hat{R}_{+i}\Phi - \hat{R}_{+i}\rho_0\hat{R}_{-i}\Phi) \\ &= \frac{i}{n}(\Omega^{-1})^{\bar{j}m}\left(D_{\bar{j}}\rho_0 D_m\Phi - D_{\bar{j}}\Phi D_m\rho_0\right) \end{aligned} \tag{100}$$

where we used (55). The density ρ_0 is proportional to the identity, so its covariant derivative is the same as the ordinary derivative; further it depends only on $r^2 = \bar{\xi}\cdot\xi$. We can then simplify the above result as

$$([\hat{\rho}_0, \hat{\Phi}])(g) = \frac{1}{n}\ (1+\bar{\xi}\cdot\xi)^2\ \frac{\partial\rho_0}{\partial r^2}\ \left(\xi^i\ D_i\Phi - \bar{\xi}^i\ D_{\bar{i}}\Phi\right) \tag{101}$$

Taking into account the scaling of the coordinates ξ we find that the large n limit of the above expression is

$$([\hat{\rho}_0, \hat{\Phi}])(g) = -\frac{i}{n}\frac{\partial\rho_0}{\partial r^2}\mathcal{L}\Phi \tag{102}$$

where

$$\mathcal{L} = i(\xi^i D_i - \bar{\xi}^i D_{\bar{i}}) \tag{103}$$

Similarly

$$([\hat{\Phi}, \hat{V}])(g) = \frac{i}{n}\frac{\partial V}{\partial r^2}\ \mathcal{L}\Phi \tag{104}$$

The leading order term in the potential term (99) as n becomes large can now be written as

$$P.E. = \sum_{l=2}^{\infty}\frac{(-i)^l}{l!}\frac{c}{n^2}\int \mathrm{tr}\left[\frac{\partial\rho_0}{\partial r^2}\ \mathcal{L}\Phi\ \underbrace{[\Phi,[\Phi,...[\Phi}_{l-2},\mathcal{L}\Phi]...]]\ \frac{\partial V}{\partial r^2}\right] + \cdots \tag{105}$$

Consider now the expression $\mathrm{tr}(G^\dagger\mathcal{L}G)^2$. We can write this as

$$\begin{aligned}\mathrm{tr}(G^\dagger\mathcal{L}G)^2 &= \mathrm{tr}\left[i\int_0^1 d\alpha e^{-i\alpha\Phi}\mathcal{L}\Phi\ e^{i\alpha\Phi}\ i\int_0^1 d\beta e^{-i\beta\Phi}\mathcal{L}\Phi\ e^{i\beta\Phi}\right] \\ &= -\mathrm{tr}\left[\int_0^1 d\alpha d\beta\mathcal{L}\Phi\ e^{i(\alpha-\beta)\Phi}\mathcal{L}\Phi\ e^{-i(\alpha-\beta)\Phi}\right] \\ &= \mathrm{tr}\left[\int_0^1 d\alpha\int_\alpha^{\alpha-1} d\gamma\sum_{l=0}^{\infty}\frac{(i\gamma)^l}{l!}\mathcal{L}\Phi\,\underbrace{[\Phi,...,[\Phi}_{l},\mathcal{L}\Phi]...]\right] \\ &= 2\sum_{l=2}^{\infty}\mathrm{tr}\left[\frac{i^l}{l!}\mathcal{L}\Phi\,\underbrace{[\Phi,...,[\Phi}_{l-2},\mathcal{L}\Phi]...]\right] \end{aligned} \tag{106}$$

There are no odd l terms in (106); the odd l terms in (105) are also zero by trace identities since the density $(\rho_0)_{\alpha\beta}$ and the potential $V_{\alpha\beta}$ are proportional to the identity matrix. Combining equations

(105), and (106) we get

$$\begin{aligned} P.E. &= \sum_{l=2}^{\infty} \frac{(-i)^l}{l!} \frac{c}{n^2} \int \mathrm{tr} \left[\frac{\partial \rho_0}{\partial r^2} \ \mathcal{L}\Phi \ [\Phi, [\Phi, ...[\Phi, \mathcal{L}\Phi]...]] \ \frac{\partial V}{\partial r^2} \right] + \cdots \\ &= \frac{c}{2n^2} \int \frac{\partial \rho_0}{\partial r^2} \ \mathrm{tr}(G^\dagger \mathcal{L} G)^2 \ \frac{\partial V}{\partial r^2} \ + \cdots \\ &= \frac{\omega c}{2n} \int \frac{\partial \rho_0}{\partial r^2} \ \mathrm{tr}(G^\dagger \mathcal{L} G)^2 \ + \ \cdots \end{aligned} \tag{107}$$

This result is for densities which behave as $\delta_{\alpha\beta}$ at large n. In the last line of (107) ρ_0 is the scalar density for the abelian background case as discussed after (91).

7. *The effective action*

We are now in a position to combine the results for the kinetic and potential energy terms. The kinetic term (74) simplifies further if we use the fact that $(\rho_0)_{\alpha\beta} \sim \delta_{\alpha\beta}$. The kinetic energy term is $K.E. = c\mathcal{S}_1(G)$, where $\mathcal{S}_1(G)$ is given in (72) and can be written as

$$\begin{aligned} K.E. &= \frac{ic}{2n} \int \mathrm{tr} G^\dagger \dot{G} \left(\hat{R}_{-i} \rho_0 \ G^\dagger \hat{R}_{+i} G - G^\dagger \hat{R}_{-i} G \ \hat{R}_{+i} \rho_0 \right) + \frac{ic}{2n} \int \rho_0 \mathrm{tr} \left(G^\dagger \dot{G} [G^\dagger \hat{R}_{+i} G, \ G^\dagger \hat{R}_{-i} G] \right) \\ &= -\frac{c}{2n} \int \frac{\partial \rho_0}{\partial r^2} \ \mathrm{tr}(G^\dagger \dot{G} \ G^\dagger \mathcal{L} G) + \frac{ic}{2n} \int \rho_0 \mathrm{tr} \left(G^\dagger \dot{G} \ [G^\dagger \hat{R}_{+i} G, \ G^\dagger \hat{R}_{-i} G] \right) \end{aligned} \tag{108}$$

In deriving the second line in (108) we used (100), (101). We can write this part of the action in a different form by using (54).

$$\begin{aligned} K.E. &= \frac{c}{2n} \int dt d\mu \ (\Omega^{-1})^{\bar{j}m} \mathrm{tr} \Big[G^{-1} \dot{G} \Big(D_{\bar{j}} \rho_0 \ G^{-1} D_m G - \ D_m \rho_0 \ G^{-1} D_{\bar{j}} G \\ &\qquad - \rho_0 \Big(\ G^{-1} D_{\bar{j}} G \ G^{-1} D_m G - G^{-1} D_m G \ G^{-1} D_{\bar{j}} G \Big) \Big) \Big] \end{aligned} \tag{109}$$

The volume element of $\mathbf{CP}^k$ given in (75) has a factor $\det \Omega$; using the relation

$$\epsilon^{i_1 i_2 ... i_{k-1} m} \epsilon^{\bar{j}_1 \bar{j}_2 ... \bar{j}_{k-1} \bar{j}} \Omega_{i_1 \bar{j}_1} \Omega_{i_2 \bar{j}_2} ... \Omega_{i_{k-1} \bar{j}_{k-1}} = (k-1)! \ (\det \Omega)(\Omega^{-1})^{\bar{j}m} \tag{110}$$

we can now write the kinetic term as

$$\begin{aligned} K.E. &= \frac{i^k kc}{2n\pi^k} \int dt \ \mathrm{tr} \Big[G^{-1} \dot{G} \Big(\partial_{\bar{j}} \rho_0 \ G^{-1} D_m G - \ \partial_m \rho_0 \ G^{-1} D_{\bar{j}} G - \ \rho_0 G^{-1} D_{\bar{j}} G \ G^{-1} D_m G \\ &\qquad + \rho_0 G^{-1} D_m G \ G^{-1} D_{\bar{j}} G \Big) \Big] \times \epsilon^{i_1 ... i_{k-1} m} \epsilon^{\bar{j}_1 ... \bar{j}_{k-1} \bar{j}} \Omega_{i_1 \bar{j}_1} ... \Omega_{i_{k-1} \bar{j}_{k-1}} d^k \xi d^k \bar{\xi} \\ &= \frac{ikc}{2n\pi} (-1)^{\frac{k(k-1)}{2}} \int dt \ \mathrm{tr} \Big[G^{-1} \dot{G} \Big(-d\rho_0 G^{-1} DG + \rho_0 G^{-1} DG \ G^{-1} DG \Big) \Big] \wedge \left(\frac{i\Omega}{\pi} \right)^{k-1} \end{aligned} \tag{111}$$

where Ω is the symplectic two-form defined in (4). The last line in (111) has been written in terms of differential forms.

Combining this with with the potential term (107), our final result for the effective action becomes

$$\begin{aligned}\mathcal{S}(G) &= -\frac{c}{2n}\int dtd\mu\ \frac{\partial\rho_0}{\partial r^2}\ \mathrm{tr}\left[\left(G^\dagger\dot{G}+\omega\ G^\dagger\mathcal{L}G\right)G^\dagger\mathcal{L}G\right] \\ &\quad +\frac{ikc}{2\pi n}(-1)^{\frac{k(k-1)}{2}}\int_{\mathcal{D}}dt\ \rho_0\mathrm{tr}\left[G^{-1}\dot{G}G^{-1}DG\ G^{-1}DG\right]\wedge\left(\frac{i\Omega}{\pi}\right)^{k-1}\end{aligned} \tag{112}$$

We have used both forms (108) and (111).

$\mathcal{S}(G)$ is a generalized chiral, gauged Wess-Zumino-Witten (WZW) action [10], [11]. Since ρ_0 is a step function as in (92), its derivative with respect to r^2 is a delta function with support at the boundary $\partial\mathcal{D}$ of the droplet, namely

$$\frac{\partial\rho_0}{\partial r^2} = -\frac{n}{M}\ \delta\left(1-\frac{nr^2}{M}\right) \tag{113}$$

Thus the first term on the right hand side of (112) is evidently a boundary term. The second term is a (gauged) Wess-Zumino term for the field G; the integration is over the droplet whose boundary is the edge $\partial\mathcal{D}$. Usually the Wess-Zumino term is written in terms of integration over a three-manifold whose boundary is the spacetime of interest; in such a representation, the Wess-Zumino term can be displayed as the integral of a local differential form. Here we see that the droplet, along with time, plays the role of this three-manifold, the radial variable of the droplet serving as the extra dimension. Of course this is not really a three-manifold, the actual dimension is $2k+1$. Only derivatives with respect to a particular spatial direction, namely $\mathcal{L}$, appear in the action. However, once the integration over the various directions are carried out the result will involve all the coordinates of the edge of the droplet for the first term. Finally, since the gauged Wess-Zumino term is the integral of a locally exact form, as shown in the next section, it should be considered as part of the boundary action; it is not a bulk term.

4 Properties of the effective action

1. *Gauge symmetry*

The effective action we have derived is a generalized gauged WZW action. The field G is a unitary $(dim\tilde{J}\times dim\tilde{J})$-matrix and it is gauged vectorially with respect to the group $SU(k)$. As we have mentioned earlier, the $U(1)$ gauge field does not appear in the covariant derivative and in the effective action.

Usually, in a gauged WZW action, the kinetic terms have gauge covariant derivatives replacing the ordinary derivatives as in (112). However, the gauging of the WZ-term is not done by replacing derivatives by covariant derivatives, rather it involves additional terms which are local boundary terms such that the anomalies of the gauge symmetry cancel out [11]. Explicitly, such a gauged WZ-term with a vector gauge symmetry is given, for a three-dimensional manifold, by

$$\Gamma_{WZ}(G,\mathcal{A}) = -\frac{1}{12\pi}\int \mathrm{tr}\left[(G^{-1}dG)^3+3d\left(\mathcal{A}\ dGG^{-1}+\mathcal{A}\ G^{-1}dG+\mathcal{A}G\mathcal{A}G^{-1}\right)\right] \tag{114}$$

The WZ-term in the effective action (112) has gauge covariant derivatives; it is of the form $\mathrm{tr}(G^{-1}DG)^3 \wedge (\Omega)^{k-1}$, with $\mathcal{A}_0 = 0$. ($\mathcal{A}_0$ is the potential corresponding to the time-direction.) Expanding out the covariant derivatives we find

$$\begin{aligned} -\frac{1}{12\pi}\mathrm{tr}(G^{-1}DG)^3 &= \Gamma_{WZ}(G,\mathcal{A}) - \frac{1}{4\pi}\mathrm{tr}\left[(dGG^{-1} + G^{-1}dG)\mathcal{F}\right] \\ &= \Gamma_{WZ}(G,\mathcal{A}) - \frac{1}{4\pi}\mathrm{tr}\left[(\dot{G}G^{-1} + G^{-1}\dot{G})\mathcal{F}\right] \\ &= \Gamma_{WZ}(G,\mathcal{A}) - \frac{1}{4\pi}\mathrm{tr}\left[(\dot{G}G^{-1} + G^{-1}\dot{G})T_a\right](-i\mathcal{F}^a) \end{aligned} \tag{115}$$

Here $\mathcal{F}$ is the field strength associated to the potential $\mathcal{A}$ and in the second line we use the fact that $\mathcal{F}$ has no time-components. The field strength $\mathcal{F}$ can be calculated from the potential as

$$\mathcal{F}^a = 2i\mathrm{tr}(t^a[t_{+i}, t_{-j}])\ E^i_k E^{\bar{j}}_{\bar{l}} d\xi^k \wedge d\bar{\xi}^{\bar{l}} \tag{116}$$

The wedge product with $d\mu(\mathbf{CP}^{k-1})$ in (115) leads to the term $\mathcal{F}^a_{\bar{l}r}(\Omega^{-1})^{\bar{l}r}$. This is verified to be zero using $(\Omega^{-1})^{\bar{l}r} = -i(E^{-1})^{\bar{l}}_{\bar{m}}(E^{-1})^r_m$ and the fact that $\sum_i \mathrm{tr}(t_a[t_{-i}, t_{+i}]) = 0$. Thus $(G^{-1}DG)^3$ does indeed reduce to the usual gauging of the WZ-term for the present case.

In summary, we have a gauged WZW model generalized to higher dimensions where the field G takes values in the unitary group $U(dim\tilde{J})$ and the gauge group is $SU(k)$, in other words a $U(dim\tilde{J})/SU(k)$ coset model.

2. *The level number of the action*

The WZ-term in (112) has the time-derivative term separated off. We can write it in a more symmetric fashion as

$$\begin{aligned} \mathcal{S}_{WZ} &= \frac{ikc}{2\pi n}(-1)^{\frac{k(k-1)}{2}} \int_{\mathcal{D}} dt\rho_0 \mathrm{tr}\left[G^{-1}\dot{G}G^{-1}DG\ G^{-1}DG\right] \wedge \left(\frac{i\Omega}{\pi}\right)^{k-1} \\ &= \frac{ikc}{6\pi n}(-1)^{\frac{k(k-1)}{2}} \int_{\mathcal{D}} \rho_0 \mathrm{tr}(G^{-1}DG)^3 \wedge \left(\frac{i\Omega}{\pi}\right)^{k-1} \\ &= \frac{ikc}{12\pi n}(-1)^{\frac{k(k-1)}{2}} \int_{\mathcal{D}} \rho_0\ 2\ \mathrm{tr}(G^{-1}DG)^3 \wedge \left(\frac{i\Omega}{\pi}\right)^{k-1} \end{aligned} \tag{117}$$

In comparing this with the usual WZ-term, we have to take account of an additional factor of $\frac{1}{2}$ which arises due to the fact that r^2, not r, is the variable of integration for the 'extra coordinate'. (The volume element will have a factor rdr; with the factor of 2 we inserted inside the integral, this will become $d(r^2)$.) Equation (117) shows that the level number of the WZ-term is given by

$$\begin{aligned} \frac{ck}{n} &= \frac{dimJ}{dim\tilde{J}}\ \frac{k}{n} \\ &= \left[\frac{(p+k-1)!}{p!(k-1)!}\right]\ \frac{p+k+jk}{p+j} \\ &\to \left[\frac{(p+k-1)!}{p!(k-1)!}\right] \end{aligned} \tag{118}$$

as $p \to \infty$. The level number is indeed an integer, as it should be, and is very large as $p \to \infty$.

Now that we have established this property, we can rewrite the action in another more transparent form. Using (92), (113), (23), (75) and the n-scaling of the coordinates we can further simplify (112) as

$$\begin{aligned}\mathcal{S}(G) &= \frac{1}{4\pi^k} M^{k-1} \int_{\partial\mathcal{D}} \mathrm{tr}\left[\left(G^\dagger \dot{G} + \omega\ G^\dagger \mathcal{L} G\right) G^\dagger \mathcal{L} G\right] \\ &\quad + (-1)^{\frac{k(k-1)}{2}} \frac{i}{4\pi} \frac{M^{k-1}}{(k-1)!} \int_{\mathcal{D}} 2\ \mathrm{tr}\left[G^\dagger \dot{G} (G^{-1} D G)^2\right] \wedge \left(\frac{i\Omega}{\pi}\right)^{k-1} \end{aligned} \tag{119}$$

The right hand side of this equation is entirely in terms of rescaled coordinates, with the radius of the droplet being unity. We have taken the limit of large n for the prefactors as well. Thus the terms displayed here are the terms which have a finite limit as $n \to \infty$; the corrections to this action are of order $1/n$.

3. *Chirality*

The effective action is a gauged WZW model and carries a notion of chirality [12]. It has the structure $\mathrm{tr}\left[\left(G^\dagger \dot{G}\ +\ \omega\ G^\dagger \mathcal{L} G\right)\ G^\dagger \mathcal{L} G\right]$; the combination $G^\dagger \dot{G}\ +\ \omega\ G^\dagger \mathcal{L} G$ indicates that there is chirality in the theory. The eventual integration over all spatial directions will give an action which has rotational symmetry, but the time-derivative will still occur in a similar combination.

4. *Relationship to other higher dimensional WZW models*

The effective action we have obtained is a higher dimensional generalization of the WZW action. Going back to (111), we write it as

$$\mathcal{S}(G) = \frac{ikc}{2\pi n} (-1)^{\frac{k(k-1)}{2}} \int \mathrm{tr}\left[G^{-1}\dot{G}\left(-d\rho_0 G^{-1} D G + \rho_0 G^{-1} D G\ G^{-1} D G\right)\right] \wedge \left(\frac{i\Omega}{\pi}\right)^{k-1} \tag{120}$$

The action density which is integrated is in the form of the exterior product of the density for the two-dimensional WZW action with Ω^{k-1}. A generalization of the WZW action to higher dimensions along these lines has been considered before [7]. The action for a k-complex dimensional Kähler manifold is

$$\mathcal{S}(G) = -\frac{1}{2\pi} \int \mathrm{tr}(G^{-1}\partial G\ G^{-1}\bar{\partial} G) \wedge \Omega^{k-1} + \frac{i}{12\pi} \int \mathrm{tr}(G^{-1} dG)^3 \wedge \Omega^{k-1} \tag{121}$$

For $k = 2$, this was obtained as the boundary action for a Kähler-Chern-Simons theory for a gauge field A with an action of the form

$$\mathcal{S} = -\frac{k}{4\pi} \int \mathrm{tr}\left(AdA + \frac{2}{3} A^3\right) \wedge \Omega\ +\ \int \mathrm{tr}(\Phi F + \bar{\Phi} F) \tag{122}$$

This action is defined on $\mathcal{M} \times [t_i, t_f]$ where $\mathcal{M}$ is a Kähler manifold and $[t_i, t_f]$ is an interval of time. Φ is a Lie algebra-valued $(2,0)$ form on $\mathcal{M}$ and a one-form for the time direction, i.e., it is of the form $\frac{1}{2}\phi_{ij} d\xi^i d\xi^j dt$. This theory has the equations of motion

$$\begin{aligned} F \wedge \Omega &= 0 \\ F_{ij} = F_{\bar{i}\bar{j}} &= 0 \end{aligned} \tag{123}$$

leading to antiselfdual fields on $\mathcal{M}$. The canonical quantization of the action (122) leads to wavefunctions of the form $\exp[i\mathcal{S}(G)]$ [7], similar to the usual connection between the Chern-Simons and WZW theories [13]. The action $\mathcal{S}(G)$ so obtained is on the boundary of $\mathcal{M} \times [t_i, t_f]$, i.e., on the two equal-time surfaces at t_i and t_f which are copies of $\mathcal{M}$, for a compact $\mathcal{M}$ with no boundary.

The difference with our present situation of Hall effect is the following. If we consider not all of $\mathcal{M}$ but a certain region $\mathcal{D}$ in $\mathcal{M}$, there is an additional boundary $\partial\mathcal{D} \times [t_i, t_f]$. The analogue of the action (121) on this boundary is what we have in the present case.

5 Discussion

We have derived the effective action, in the limit of large number of fermions, for quantum Hall droplets in $\mathbf{CP}^k = SU(k+1)/U(k)$ in the presence of abelian ($U(1)$) and nonabelian ($SU(k)$) background magnetic fields. This result, which is the main result of this paper, is summarized in (112) and (119). The action describes a higher dimensional gauged WZW model. With the $SU(k)$ nonabelian background field, the states of the lowest Landau level belong to an $SU(k+1)$ representation J, with a set of lowest weight states forming a representation $\tilde{J}$ of $SU(k)$. The effective action is for a unitary matrix field G which is an element of $U(dim\tilde{J})$, which is coupled to the background $SU(k)$ field in a left-right symmetric or vectorial way. There is an $SU(k)$ gauge invariance to take account of this coupling to the $SU(k)$ gauge field. Thus, effectively, the field space is the coset $U(dim\tilde{J})/SU(k)$.

The effective action we have derived is very similar to the gauged version of the Kähler WZW model used in the context of higher dimensional conformal field theories [7]. The WZ-term of this action is defined on the $(2k+1)$-dimensional manifold $\mathcal{D} \times \mathbf{R}$, corresponding to the droplet $\mathcal{D}$ and time. The rest of the action is defined on $\partial\mathcal{D} \times \mathbf{R}$, corresponding to the boundary of the droplet and time. The radial variable for the droplet plays the role of the extra dimension in expressing the WZ-term as the integral of a local differential form. Further, the action involves the spatial derivative along a particular tangential direction on the boundary of the droplet, which is eventually integrated over. There is also a certain chirality property due to the manner in which the space and time derivatives appear. Finally, as expected, for an abelian background, the action reduces to the chiral $U(1)$ bosonic action given in [5].

The effective action (119) has a very interesting current algebra structure which will have implications on the nature of the spectrum of edge excitations. This facet of the problem and related issues will be analyzed in more detail elsewhere.

The WZW theory is well known to lead to conformal field theories in two dimensions; in particular it is a fixed point under the renormalization flow. Given that we find a generalized version of the WZW theory, it is natural to ask how it changes under renormalization. We have not carried out a full study of this question, but the following remark may be interesting in this context. The variational problem for the action (1) gives the full quantum dynamics. Apart from assuming a Hartree-Fock type factorization of the many-body wavefunctions in terms of one-particle wavefunctions, which was explained in [5], we have not made any other approximation. Thus the effective action we derived is the exact quantum action in the large n limit as far as states in the

lowest Landau level are concerned. Contributions from virtual transitions to higher Landau levels have not been calculated.

For a Fermi liquid in dimensions higher than one, expansion around the Fermi surface has been used to rewrite the low energy excitations in terms of a set of one-dimensional models which can be bosonized. What we have done here is similar; our result may be viewed as a partial bosonization which is adequate for the low energy excitations.

Finally, we note that while quantum Hall effect and the droplet picture may serve as a physical picture for this class of theories, they can be given an independent formulation. The Landau levels and the effective field G are sections of appropriate H-bundles on a G/H space. The kinetic term of the effective action is an analogue of the Kirillov form. Thus the class of theories we are discussing may be viewed in a purely geometric way. It is also clear that these theories may have some relevance to the dynamics of droplets of incompressible fluids in general.

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